

All questions may be attempted but only marks obtained on the best **four** solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

1. (a) Let k be a field. Give the definition of an irreducible polynomial in $k[x]$, and state the unique factorization theorem for polynomials.
- (b) Factorize $f(x) = x^3 + 2x^2 + 2x + 1$ into irreducibles over \mathbb{R} , \mathbb{C} and \mathbb{F}_3 , justifying why your factors are irreducible in each case.
- (c) Does $f(x) = x^{2013} - 78x^4 + x - 1$ have a root in \mathbb{R} ? Justify your answer.
- (d) Define the greatest common divisor of two polynomials $f, g \in k[x]$, and say what it means for f and g to be coprime. Use Bézout's identity to show that if $f|gh$ where f and g are coprime, then $f|h$.

2. Let V be a vector space over k and let $T : V \rightarrow V$ be a linear map.
 - (a) Say what is meant by T being diagonalizable. Define the minimal polynomial of T and state a criterion for the diagonalizability of T involving the minimal polynomial.
 - (b) Define the generalized eigenspace $V_i(\lambda)$ associated to an eigenvalue λ of T , and show that
 - (i) $V_i(\lambda) \subset V_{i+1}(\lambda)$;
 - (ii) $T(V_i(\lambda)) \subset V_i(\lambda)$.
 - (c) For each of the following cases, find the Jordan normal form of T
 - (i) $\text{ch}_T(x) = (x - 1)^3$, $m_T(x) = (x - 1)^2$;
 - (ii) $\text{ch}_T(x) = (x - 1)^4$, $\dim(V_1(1)) = 3$;
 - (iii) $\text{ch}_T(x) = (x - 1)^9$, $\dim(V_1(1)) = 5$, $\dim(\text{im}((T - \text{id})^2)) = 1$.

3. Let $V = \mathbb{R}[x]_3$ be the vector space of real polynomials of degree ≤ 3 . Let $T : V \rightarrow V$ be the linear map

$$T = 3\text{Id} + D^2,$$

where Id is the identity map and $D : V \rightarrow V$ is the linear map taking a polynomial f to its derivative f' .

- Choose a basis \mathcal{B} for V and compute the matrix representative $[T]_{\mathcal{B}}$ for T .
- Find the minimal polynomial of T .
- For each eigenvalue λ of T , find the generalized eigenspaces $V_i(\lambda)$ for each i .
- Find the Jordan Normal Form of T .
- Find a Jordan basis for V .

4. (a) Define the dual space V^* of a vector space V over a field k (including the definition of addition and scalar multiplication in V^*).
- (b) Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for a finite dimensional vector space V . Say what is meant by the dual basis $\mathcal{B}^* = \{v_1^*, \dots, v_n^*\}$.
- (c) Let $V = \mathbb{R}[x]_2$ be the vector space of real polynomials of degree ≤ 2 . Define three linear forms $\alpha_i : V \rightarrow \mathbb{R}$ ($i = 1, 2, 3$) by

$$\alpha_1(f) = \int_0^1 f(x)dx, \quad \alpha_2(f) = \int_0^2 f(x)dx, \quad \alpha_3(f) = \int_0^3 f(x)dx.$$

By finding polynomials $f_1, f_2, f_3 \in \mathbb{R}[x]_2$ such that $f_1^* = \alpha_1$, $f_2^* = \alpha_2$ and $f_3^* = \alpha_3$, show that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for V^* .

5. (a) Let V be a finite dimensional vector space over a field k in which $1 + 1 \neq 0$. Let f be a symmetric bilinear form on V with associated quadratic form q .
- (i) Define the orthogonal complement S^\perp of a subset $S \subset V$.
 - (ii) Prove that if a vector $v \in V$ satisfies $q(v) \neq 0$, then

$$V = \text{span}\{v\} \oplus \{v\}^\perp.$$

- (iii) Prove there exists a basis for V which is orthogonal with respect to f .
- (b) Determine the real canonical form, rank and signature of the following quadratic form

$$q \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x^2 + 4xz + 2yz$$

If f is the associated symmetric bilinear form to q , does f define an inner product on \mathbb{R}^3 ? Justify your answer.

6. Let $(V, \langle -, - \rangle)$ be an inner product space.

- (a) State and prove the Cauchy-Schwartz inequality for V .
- (b) Say what it means for a linear map $T : V \rightarrow V$ to be self-adjoint. Give an example of a self-adjoint linear map on
 - (i) a real inner product space;
 - (ii) a complex inner product space.

Be sure to specify the inner product.

- (c) Show that all eigenvalues of a self-adjoint linear map are real.
- (d) If $T : V \rightarrow V$ is an isometry, what can you say about the eigenvalues of T ?